hep-th/9603081
USITP-96-03, UUITP-2/96
March 1996

D-particle Dynamics and Bound States

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Abstract

We study the low energy effective theory describing the dynamics of D-particles. This corresponds to the quantum mechanical system obtained by dimensional reduction of $9+1$ dimensional supersymmetric Yang-Mills theory to $0+1$ dimensions and can be interpreted as the non relativistic limit of the Born-Infeld action. We study the system of two like-charged D-particles and find evidence for the existence of non-BPS states whose mass grows like $\lambda^{1/3}$ over the BPS mass. We give a string interpretation of this phenomenon in terms of a linear potential generated by strings stretching from the two D-particles. Some comments on the possible relations to black hole entropy and eleven dimensional supergravity are also given.

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1 Introduction

The last year has seen a tremendous activity in the subject of nonperturbative string theory. Not only have fascinating dualities between various string theories been proposed, but also we have obtained powerful tools to investigate string solitons, collectively known as p-branes[4], see e.g. [1, 2]. In particular, those solitons arising in type IIA and IIB string theory and carrying Ramond-Ramond (RR) charges can, according to [3, 4], be described using Dirichlet-branes (D-branes). The D-branes allow us to use string theory tools to study objects that previously have been accessible only in the field theory limit.

An exciting application is the study of black holes. Several low dimensional examples have been discussed in detail, see e.g. [5, 6, 7, 8, 9, 10]. By wrapping higher dimensional p-branes around compact cycles and considering bound states of such objects, black holes in the compactified theory can be constructed. These studies have been particularly successful in cases where the string coupling is constant over spacetime or at least remains small near the horizon. This is the case for some dyonic black holes with both magnetic and electric charges. Using the D-brane technology of [3, 4] one can then reliably compute the number of states given their charges and masses. As a remarkable illustration of the power of string theory it is then found that the corresponding entropy agrees with the prediction of Bekenstein-Hawking.

In this paper we will study D-particles (0-branes) in ten dimensional type IIA string theory. In e.g. [11] such solitonic objects have been constructed using the effective field theory equations. In particular there are extremal objects with vanishing horizon area (and entropy) with mass $M = Q/\lambda$ in the string metric[5]. Q is the RR charge and λ the asymptotic value of the string coupling. Since the theory also contains 6-branes that are electromagnetic duals to the D-particles, the analysis of Dirac implies that Q must be quantized. In two remarkable papers, [12, 13] (see also [14, 15]), it has been argued that these objects also have an eleven dimensional explanation. They correspond to Kaluza-Klein states of eleven dimensional supergravity compactified on a circle with radius $R \sim \lambda^{2/3}$. Further results on higher dimensional D-branes in type IIA can be found in [16, 17].

The extremal black holes have properties that identify them as BPS states. They break half the supersymmetries and are also stable according to the classical Bekenstein-Hawking analysis. The BPS states are extremely important objects since it is possible to make exact statements about their properties. However, from a physical point of view, the non-BPS states are perhaps even more interesting. They are not expected to be stable and given the Bekenstein-Hawking thermodynamic argument they should decay through Hawking radiation. Excited, non-BPS D-particles were briefly discussed in [6, 18]. For a single D-particle these states are described by open strings with both ends attached to the D-particle. The only possible excitations (in contrast to higher dimensional p-branes) are massive modes

[4] p is the dimension of the soliton: $p = 0$ for particle-like objects, $p = 1$ for strings etc.

[5] Throughout the paper we use dimensionless units by setting $\alpha' = 1$.

of the string. For D-particles of higher charge, however, there are other possibilities to construct non-BPS states. This is what will be the main subject of the paper.

The paper is organized as follows. In section two we consider the derivation of effective theories for D-branes. The appearance of the Born-Infeld action is discussed and it is argued that it is consistent to limit oneself to the study of the Yang-Mills theory describing its non relativistic limit. In section three we study in detail the supersymmetric quantum mechanical problem describing a system of two bound D-particles. We treat the problem in the Born-Oppenheimer approximation and show the existence of non BPS states of mass $2/\lambda + \lambda^{1/3}\epsilon$, where $2/\lambda$ is the BPS mass for $Q = 2$ and ϵ the eigenvalue of a one dimensional Schrödinger equation independent on λ. In section four we present the string interpretation of the result. We show how each bound state can be interpreted as a particular configuration of straight open strings stretching between the D-particles. In section five we conclude with some speculation about the possible role of eleven dimensional supergravity.

2 p-branes from D-branes

According to the D-brane prescription a p-brane with RR charges can be described by an open string theory where the string end points are restricted to lie on the D-brane. The effective field theory for this open string theory living on the D-brane can be obtained by standard techniques, e.g [19, 20]. The open strings have Neumann boundary conditions for p of the spatial coordinates and therefore couple to gauge fields within the D-brane. For a single D-brane we simply get a $U(1)$ gauge group. What kind of gauge theory do we obtain? According to [20], see also [21, 22], we do not find the familiar Yang-Mills theory but rather a Born-Infeld action. The bosonic part of the effective action for a D-brane is

$$S_p = \int d^{p+1}\sigma e^{-\phi}\mathrm{Tr}\sqrt{-\det(G+F)} \tag{1}$$

where G is the induced metric on the D-brane and F the field strength of the $U(1)$ gauge field coupling to the open string. For a D-particle this reduces to

$$S_0 = \frac{1}{\lambda}\int dt\sqrt{-\det G} = \frac{1}{\lambda}\int dt\sqrt{1-v^2} \tag{2}$$

for $e^{\phi} = \lambda$. The action is simply the action for a relativistic particle with mass $M = 1/\lambda$.

Another way to obtain this result is to start with the ten dimensional open string theory, i.e. a 9-brane. In this case all string coordinates have Neumann boundary conditions and couple to electromagnetic fields in the ten dimensional space time. To obtain the action for a $p < 9$-brane we must impose Dirichlet boundary conditions [23] in $9 - p$ of the spatial directions. This can effectively be taken care of by a dimensional reduction in the directions where we want Dirichlet

boundary conditions. We then need to reinterpret the corresponding components of the gauge field as the new transversal coordinates. If, in the ten dimensional action, we restrict ourselves to gauge potentials depending only on time, $A_i = A_i(t)$, the fully dimensionally reduced action is simply $\frac{1}{\lambda}\int dt\sqrt{1-\dot{A}^2}$. A_i should now be thought of as the position of the D-particle. From this point of view it becomes clear that the Yang-Mills approximation to Born-Infeld is equivalent to considering nonrelativistic D-particles. In fact, within the nonrelativistic approximation we could as well have started with the ten dimensional Yang-Mills limit of the Born-Infeld action. This is what we will do in the following section.

The parallel with special relativity was indeed the motivation for Born and Infeld to introduce their action in the thirties [24]. As pointed out in [25], the form of the Born-Infeld action is fixed through Neuman-Dirichlet duality.

For n superimposed D-branes the gauge group to use, according to [26], is $U(n)$. If we factor out a $U(1)$ (it just corresponds to a common spatial translation of all the D-particles) the relevant gauge group is instead $SU(n)$. However, the non-abelian gauge bosons get masses if the superimposed D-branes are pulled apart, that are proportional to the string tension and the distance between the D-branes. The physical picture of a W-boson is that of a string stretching between the D-branes. We will find a nice illustration of this phenomenon in the calculations of the following sections. Eventually, when the D-particles are far enough apart, the theory becomes that of n independent $U(1)$'s, one for each D-brane describing their positions in space-time.

Presumably the full theory is some non-abelian supersymmetric generalization of Born-Infeld dimensionally reduced to zero dimensions. We do not know how to write down such a theory. Instead we will follow [26], see also [27, 28], and start with $N=1$ SUSY Yang-Mills in ten dimensions with gauge group $SU(n)$. In particular we will study the case of $SU(2)$. From our point of view, the use of Yang-Mills rather than Born-Infeld corresponds to a nonrelativistic approximation as explained above[6]. Further on, when we discuss the presence of bound states this means that we must restrict ourselves to only light excitations as compared to the mass of the D-particles.

3 The study of the effective theory

In this section we derive and study the effective theory describing the interaction of two D-particles of equal charge. We derive the Schrödinger equation by dimensional reduction and study it using the Born-Oppenheimer approximation. We calculate the spectrum and the degeneracies and find the existence of bound states above the

[6]Strictly speaking, the identification of non-abelian Yang-Mills theory as giving the non-relativistic approximation of the action for several D-branes is threatened by higher order non-abelian terms [29, 30]. We thank A. Tseytlin for bringing this fact to our attention. Such terms do not affect our conclusions for small coupling λ.

BPS ground state.

3.1 Notation and conventions.

We start by considering $N = 1$ supersymmetric Yang-Mills theory in $9+1$ dimensional Minkowski space with metric $g_{\mu\nu} = \mathrm{diag}(+1, -1, \cdots, -1)$. The field content is the gauge potential A^a_μ and a Majorana-Weyl spinor Ψ^a in the adjoint representation of the gauge group. We fix the gauge group to be $SU(2)$, and denote the structure constants by ϵ^{abc}. The 32×32 dimensional Dirac matrices Γ_μ $\mu = 0, \cdots, 9$, satisfying $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$, are written as

$$\begin{aligned}\Gamma_0 &\equiv \Gamma^0 = \mathbf{1} \otimes \sigma_2 \\ \Gamma_i &\equiv -\Gamma^i = \gamma_i \otimes i\sigma_1 \\ \Gamma_S &\equiv \Gamma^0 \cdots \Gamma^9 = \mathbf{1} \otimes \sigma_3,\end{aligned} \tag{3}$$

where γ_i are 9 real symmetric 16×16 dimensional matrices satisfying $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, $\mathbf{1}$ is the 16 dimensional identity matrix and $\sigma_{1,2,3}$ the usual Pauli matrices. The first 8 matrices γ can be identified with the Dirac matrices of spin(8) and the last with the 8 dimensional chirality $\gamma_9 = \gamma_1 \cdots \gamma_8$. The action is[7]

$$S = \frac{1}{2\lambda} \int d^{10}x \, \left(-\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \frac{i}{2} \bar{\Psi}^a \Gamma^\mu D_\mu \Psi^a \right), \tag{4}$$

where

$$\begin{aligned}F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu \\ D_\mu \Psi^a &= \partial_\mu \Psi^a + \epsilon^{abc} A^b_\mu \Psi^c\end{aligned} \tag{5}$$

are the field strength and the covariant derivative respectively. The supersymmetry transformation that leaves the action invariant can be written by introducing a constant anticommuting Majorana-Weyl spinor ϵ

$$\begin{aligned}\delta A^a_\mu &= i\bar{\epsilon}\Gamma_\mu \Psi^a \\ \delta \Psi^a &= \frac{1}{2} F^a_{\mu\nu} \Gamma^{\mu\nu} \epsilon\end{aligned} \tag{6}$$

where $\Gamma^{\mu\nu} = [\Gamma^\mu, \Gamma^\nu]/2$ and the same notation will be used below for the γ_i's. The transformations (6) yield the supercurrent

$$J^\rho = \frac{i}{2} \Gamma^{\mu\nu} \Gamma^\rho F^a_{\mu\nu} \Psi^a. \tag{7}$$

[7] In ordinary Yang-Mills theory $2\lambda = g^2$, with g the Yang-Mills coupling. The factor of 2 comes from considering the reduced mass of the two D-particles, each of mass $1/\lambda$.

3.2 Dimensional reduction

The dimensional reduction [31] of (4) all the way down to $0+1$ dimensions (supersymmetric quantum mechanics [32, 33, 34]) can be easily performed by letting all the fields be independent of all space components $x_1, \cdots, x_9$. It is also more convenient to work with the 16 components spinors ψ^a defined through

$$\Psi^a = \sqrt{2\lambda}\, \psi^a \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{8}$$

rather than Ψ^a. Representing the time derivative by a dot, the various components of the field strength and the covariant derivative are

$$\begin{aligned} F^a_{0i} &= \dot{A}^a_i + \epsilon^{abc} A^b_0 A^c_i \\ F^a_{ij} &= \epsilon^{abc} A^b_i A^c_j \\ D_0 \psi^a &= \dot{\psi}^a + \epsilon^{abc} A^b_0 \psi^c \\ D_i \psi^a &= \epsilon^{abc} A^b_i \psi^c. \end{aligned} \tag{9}$$

Substituting in (4) and dropping the volume term we obtain

$$\begin{aligned} S &= \int dt \left[\frac{1}{2\lambda} \left(\frac{1}{2} \dot{A}^{a2}_i + \epsilon^{abc} \dot{A}^a_i A^b_0 A^c_i + \frac{1}{2} \left(\epsilon^{abc} A^b_0 A^c_i \right)^2 - \frac{1}{4} \left(\epsilon^{abc} A^b_i A^c_j \right)^2 \right) \right. \\ &+ \left. \frac{i}{2} \psi^a \dot{\psi}^a + \frac{i}{2} \epsilon^{abc} \psi^a A^b_0 \psi^c + \frac{i}{2} \epsilon^{abc} A^a_i \psi^b \gamma_i \psi^c \right]. \end{aligned} \tag{10}$$

All space indices are now "internal" indices of this quantum system and will always be written downstairs, with the convention that two repeated indices are summed with the metric $+\delta_{ij}$. The transpose sign ^{T} on the spinor to the left should also always be understood. As in any gauge theory, the A_0 component is an auxiliary field enforcing the Gauss law

$$G^a(t) = \frac{\delta S}{\delta A^a_0(t)} = 0. \tag{11}$$

We shall work in the "temporal gauge" $A^a_0 \equiv 0$ and denote by E^a_i the momentum conjugate to the A^a_i. We then quantize the theory by introducing the canonical commutation and anti-commutation relations

$$\begin{aligned} [E^a_i, A^b_j] &= -i \delta_{ij} \delta^{ab} \\ \{\psi^a_\alpha, \psi^b_\beta\} &= \delta^{ab} \delta_{\alpha\beta}. \end{aligned} \tag{12}$$

In dimensionally reducing the action from $9+1$ to $0+1$ dimensions, we have gone from $N=1$ to $N=16$ real supersymmetries, whose generators we denote by Q_α.

In the temporal gauge, the Gauss law G^a, the supercharges Q_α and the Hamiltonian H read

$$G^a = \epsilon^{abc} A^b_i E^c_i - \frac{i}{2} \epsilon^{abc} \psi^b \psi^c, \tag{13}$$

$$Q_\alpha = \sqrt{2\lambda}\gamma_{i\alpha\beta}\psi^a_\beta E^a_i - \frac{1}{2\sqrt{2\lambda}}\epsilon^{abc}\gamma_{ij\alpha\beta}\psi^c_\beta A^a_i A^b_j, \tag{14}$$

and

$$H = \lambda E^{a2}_i - \frac{1}{2}A^a_i K^a_i + \frac{1}{8\lambda}\left(\epsilon^{abc}A^b_i A^c_j\right)^2. \tag{15}$$

In the expression for the Hamiltonian we have lumped all the dependence on the fermionic degrees of freedom into the operator

$$K^a_i = i\epsilon^{abc}\psi^b\gamma_i\psi^c. \tag{16}$$

The algebra satisfied by the above operators is

$$\begin{aligned} [G^a, G^b] &= i\epsilon^{abc}G^c \\ [G^a, Q_\alpha] &= 0 \\ \{Q_\alpha, Q_\beta\} &= 2\delta_{\alpha\beta}H - 2\gamma_{i\alpha\beta}A^a_i G^a, \end{aligned} \tag{17}$$

and thus the supersymmetry algebra is satisfied only weakly.

3.3 General comments on the quantization procedure

A physical way to study the system is to use the Born-Oppenheimer approximation, treating the fast oscillations orthogonal to the classical vacuum as the "electrons" and then restricting oneself on the minimum of the bosonic potential (the "nuclei"). Schematically, we shall write the full wave function $\mathbf{\Psi}$ as

$$\mathbf{\Psi} = \mathbf{\Xi}(\mathrm{slow}) \otimes \mathbf{\Phi}(\mathrm{fast}, \mathit{slow}), \tag{18}$$

where $\mathbf{\Xi}$ is the wave function depending only on the coordinates of the vacuum and $\mathbf{\Phi}$ is the wave function along the non degenerate directions, parametrized by the slow coordinates of the vacuum. Unfortunately, as we shall see, the approximation breaks down near the origin $A = 0$ and this prevents one from making a rigorous statement about the existence of the zero energy state. It does however allow one to make some non trivial consistency checks and also, perhaps more interestingly, to study the existence of excited states, where the presence of a linearly binding potential at infinity cures the problems at the origin by making it possible to find smooth and normalizable wave functions within this approximation.

The reason why these excited states arise is that, in spite of the fact that the system has flat directions, the potential away from these direction becomes steeper and steeper as one moves to the large field region. In other words, a classical particle rolling along the classical vacuum sees the walls of the potential "closing in". This essentially prevents a wave packet, (or quantum particle) from spreading to infinity. This effect is captured by the Born-Oppenheimer approximation by showing that the quantization along the non-flat directions gives rise to a linear potential that binds the particle. This effect is of course not present for the ground state, where

the bosonic and fermionic contributions cancel out, the quantum mechanical analog of the familiar field theoretical statement.

Let us also mention that, if the ground state exists and is unique, then, by virtue of the commutation relations among (13) and (14) it is guaranteed to be gauge invariant. This is so because if one defines the unitary operator $U(\theta) = \exp(i\theta^a G^a)$ for 3 arbitrary real numbers θ^a then, if $Q_\alpha \mathbf{\Psi} = 0$, it is also $Q_\alpha U(\theta)\mathbf{\Psi} \equiv U(\theta) Q_\alpha \mathbf{\Psi} = 0$. But, since the ground state is unique, $\mathbf{\Psi} = U(\theta)\mathbf{\Psi}$ (up to an irrelevant phase), i.e. $\mathbf{\Psi}$ gauge invariant. However, the gauge invariance of the excited states is not given by this simple argument but it will be imposed during quantization and it will pose some restrictions on the degeneracy of the spectrum.

Usually, the presence of the first class constraints introduces another slight complication; a gauge invariant state is, in general, not normalizable on the full configuration space and normalizability should only be imposed after having factored out the gauge directions. Such problem does not arise in our case because, after having fixed the temporal gauge, the only remnant of "gauge" invariance is a global $SU(2)$ transformation of finite volume.

3.4 The classical ground state

We begin our analysis with a more thorough investigation of the classical ground state. We are dealing with a supersymmetric quantum mechanical system described by 27 bosonic coordinates A^a_i, $a = 1, 2, 3$, $i = 1, \cdots, 9$ and 48 fermionic ones ψ^a_α, $a = 1, 2, 3$, $\alpha = 1, \cdots, 16$[8]. It is easy to see that the bosonic potential

$$V = \frac{1}{8\lambda}\left(\epsilon^{abc} A^b_i A^c_j\right)^2 \tag{19}$$

vanishes on what can be identified as an 11 dimensional cone[9]. To see this, consider the mapping

$$\mathcal{C} \equiv \left(\mathbf{R}^9 \setminus \{\mathbf{0}\}\right) \times \mathbf{S}^2 \to \mathbf{R}^{27}, \tag{20}$$

defined by

$$A^a_i = \lambda_i n^a, \quad \lambda_i \in \mathbf{R}, \quad n^a n^a = 1 \quad r^2 \equiv \lambda_i^2 \neq 0. \tag{21}$$

Eq. (21) is obviously a minimum of (19). Moreover, the Hessian of (19) at a generic point of (21) is

$$\frac{\partial^2 V}{\partial A^a_i \partial A^b_j} = \frac{1}{2\lambda} r^2 \left(\delta_{ij} - \frac{\lambda_i \lambda_j}{r^2}\right)\left(\delta^{ab} - n^a n^b\right), \tag{22}$$

which has 16 non degenerate directions and 11 degenerate ones for $r \neq 0$; (for $r = 0$ all directions are degenerate and that represents the apex of the cone).

[8] Recall that supersymmetry is only realized on shell, after eliminating 3 of the bosonic coordinates.

[9] In order to avoid confusion, we should like to stress that this includes some gauge equivalent configurations, it is not the "moduli" space of the theory, which is simply $\mathbf{R}^9$.

Hence, away from $r = 0$ the map (20) is regular and describes a two-to-one map of $(\mathbf{R}^9 \setminus \{\mathbf{0}\}) \times S^2$ into $\mathbf{R}^{27}$. The reason why the map is two-to-one is because the points (λ_i, n^a) and $(-\lambda_i, -n^a)$ are mapped to the same point of $\mathbf{R}^{27}$. One could mod out this $\mathbf{Z}_2$ factor from the sphere but this is not necessary for our purposes, as long as we remember that $\lambda_i \to -\lambda_i$ is a gauge transformation. Notice that the situation generalizes to an arbitrary group G in a straightforward way. If $d = \dim(G)$, $r = \mathrm{rank}(G)$, then the bosonic vacuum is a $8r + d$ dimensional cone, leaving a total of $8(d - r)$ non degenerate directions in the full space of potentials $\mathbf{R}^{9d}$.

3.5 Quantization along the non-degenerate directions.

Let us first fix a point on $\mathcal{C}$ (away from the apex) and consider the quantization of the "fast" modes, i.e. the non degenerate directions of the Hessian, in the same spirit as in the Born-Oppenheimer approximation. All these points are in fact equivalent and, to fix the ideas, we shall take the point on $\mathcal{C}$ described by $A^3_9 = r$ and $A^a_i = 0$ otherwise. By constructing the generic tangent vector to $\mathcal{C}$, it is easy to see that the 16 orthogonal directions are $\delta A^a_i \equiv a^a_i$, non zero only for $a = 1, 2$ and $i = 1, \cdots, 8$. Denoting by e^a_i their momenta, the Hamiltonian for the fast modes can be written near that point as

$$H_{\text{fast}} = \sum_{a=1,2\; i=1\cdots 8} (\lambda e^{a2}_i + \frac{1}{4\lambda} r^2 a^{a2}_i) - \frac{r}{2} K^3_9 \equiv H_B + H_F, \tag{23}$$

where K^3_9 is defined as in (16).

The first thing to notice is that the lowest eigenvalue of (23) is identically zero. This comes from the fact that the zero point energy $16 \times (r/2)$ of the 16 bosonic oscillators in H_B is cancelled by the lowest eigenvalue of the fermionic part H_F. We will now study the spectrum in more details, taking care of gauge invariance and will recover this fact as a simple consequence. For now it only needs to be said that, away from the apex of $\mathcal{C}$ the quantization of the fast modes yields the expected result: there is a unique normalizable wave state $\mathbf{\Phi}$ such that $H_{\text{fast}}\mathbf{\Phi} = 0$.

In order to study the spectrum in more details, we have to understand how gauge invariance acts on the fast modes described by (23). Two of the generators of $SU(2)$ act on the coordinates of $\mathcal{C}$ and therefore should not be considered here where the point on the vacuum has to be held fixed. They actually only determine how the fast modes at one point in the vacuum are related to fast modes at neighbouring points. But the last generator, generically $G = n^a G^a$, ($G = G^3$ in our case) acts in the perpendicular direction to the vacuum and in the Born-Oppenheimer approximation we must project out all those modes that do not satisfy $G\mathbf{\Phi} = 0$, where $\mathbf{\Phi}$ is the wave function of the fast modes.

Let us begin with the bosonic part. The bosonic part G_B of G is the sum of 8

angular momentum operators, all commuting with (23)

$$G_B = \sum_{j=1,\cdots 8} L_j^3, \qquad L_j^3 = -i\epsilon^{3bc} a_j^b \frac{\partial}{\partial a_j^c} \quad \text{(no sum over } j \text{ here)}. \tag{24}$$

This means that the best way to think of the bosonic part of (23) is as the sum of 8 two dimensional harmonic oscillators $H_B = \sum H_i$, each characterized by two quantum numbers (N_i, m_i), $m_i = 0, \pm 2, \cdots \pm N_i$ if N_i even, $m_i = \pm 1, \pm 3, \cdots \pm N_i$ if N_i odd. The complete bosonic part is therefore labelled by 16 quantum numbers $\mathbf{\Phi}_{N_1,m_1,\cdots,N_8,m_8}$, with total bosonic energy $E_B = r(N_B + 8)$; $N_B = N_1 + \cdots + N_8$ and total angular momentum $G_B = m_1 + \cdots + m_8$. It is then a straightforward combinatorial problem to show that the degeneracy of a state with such total energy and angular momentum is:

$$d_B(N_B, G_B) = \binom{(N_B + G_B)/2 + 7}{7} \binom{(N_B - G_B)/2 + 7}{7}. \tag{25}$$

(Note that $N_B \pm G_B$ is always an even integer.)

One might think that gauge invariance requires $G_B = 0$ but one has also to take into account the fermionic modes as well, and keep those states for which $G = G_B + G_F = 0$. The fermionic modes can be treated as follows: consider the creation and annihilation operators

$$\begin{aligned} a_\alpha &= \begin{cases} \frac{1}{\sqrt{2}} \Omega_{\alpha\beta} \left(\psi_\beta^1 + i\psi_\beta^2\right) & \text{for } \alpha = 1, \cdots 8 \\ \frac{1}{\sqrt{2}} \Omega_{\alpha\beta} \left(\psi_\beta^1 - i\psi_\beta^2\right) & \text{for } \alpha = 9, \cdots 16 \end{cases} \\ a_\alpha^\dagger &= (a_\alpha)^\dagger \end{aligned} \tag{26}$$

where Ω is an orthogonal matrix that diagonalizes $\lambda_i \gamma_i$ ($= r\gamma_9$ in our case) as $\Omega(\lambda_i\gamma_i)\Omega^T = r\hat{\delta}$, the diagonal matrix $\hat{\delta}$ having matrix elements $\hat{\delta}_{\alpha\beta} = +\delta_{\alpha\beta}$ for $\alpha \le 8$ and $\hat{\delta}_{\alpha\beta} = -\delta_{\alpha\beta}$ for $\alpha > 8$.

In terms of a and $a^\dagger$, the fermionic Hamiltonian and the fermionic Gauss law are:

$$H_F = r(a_\alpha^\dagger a_\alpha - 8) \equiv r(N_F - 8); \quad G_F = -\hat{\delta}_{\alpha\beta} a_\alpha^\dagger a_\beta. \tag{27}$$

The vacuum $|0>$ is manifestly gauge invariant and its zero energy cancels the bosonic zero energy as promised. All other states can be constructed by acting with $a^\dagger$'s on the vacuum. They can be labeled in terms of the two quantum numbers $N_F = 0, \cdots, 16$ and $G_F = 0, \pm 2, \cdots \pm N_F$ for N_F even and $G_F = \pm 1, \pm 3, \cdots \pm N_F$ for N_F odd. Their degeneracy can also be easily computed as:

$$d_F(N_F, G_F) = \binom{8}{(N_F + G_F)/2} \binom{8}{(N_F - G_F)/2}. \tag{28}$$

It is now possible to combine these two results and to see that the gauge invariant fast modes are characterized by an overall energy $E = E_B + E_F = rN$, N an even

integer and that the overall degeneracy of the gauge invariant sector is

$$d(N) = \sum_{M=1,\cdots 16} \sum_{\text{allowed } G'} d_B(N-M, G') d_F(M, -G') \approx \frac{197 \cdot 199}{2^{14} \cdot 7!^2} N^{14}. \qquad (29)$$

The first exact values are $d(0) = 1$, $d(2) = 192$, $d(4) = 11280$ and the asymptotic formula is within 1% for $N > 100$. The degeneracy is of course only power law, as expected for a finite number of oscillators and should be compared with the degeneracy of an unconstrained system of 16 oscillators that grows like N^{15} instead of N^{14}. The energy rN is the linear potential generated by the (unexcited) strings stretching between the two D-particles and is responsible (for $N \neq 0$) for the existence of the excited bound states to be discussed in the next section.

3.6 Quantization along the flat directions

The Hamiltonian H_{slow} on the vacuum is essentially the Laplacian on the cone plus the linear effective potential found in the previous section. The Laplacian can be computed by noticing that the induced metric on the vacuum (21) is

$$ds^2 = dA_i^{a2} = d\lambda_i^2 + \lambda_i^2 dn^{a2} = dr^2 + r^2 d\Omega_8 + r^2 d\Omega_2, \qquad (30)$$

where $d\Omega_8$ and $d\Omega_2$ are the metrics on the unit spheres. Denoting by L_8 and L_2 the corresponding angular momenta, we obtain [10]:

$$H_{\text{slow}} = -\frac{\lambda}{r^{10}} \partial_r r^{10} \partial_r + \frac{\lambda}{r^2} \left(L_8^2 + L_2^2 \right) + Nr. \qquad (31)$$

This Hamiltonian can be simplified by recalling the familiar result that the eigenvalues of the two angular momentum operators are given by the quadratic Casimirs of $SO(3)$ and $SO(9)$ in the totally symmetric representations, characterized by a single integer l_2 and l_8. Also, we can rewrite the radial operator as

$$-\frac{1}{r^{10}} \partial_r r^{10} \partial_r = -\frac{1}{r^5} \partial_r^2 r^5 + \frac{20}{r^2}, \qquad (32)$$

and redefine the radial wave function as $y(r) = r^5 \mathcal{R}(r)$ to obtain the one dimensional problem:

$$-\lambda \frac{d^2}{dr^2} y(r) + \left(\lambda \frac{20 + l_8(l_8+7) + l_2(l_2+1)}{r^2} + Nr \right) y(r) = E_{\text{tot}} y(r). \qquad (33)$$

Before continuing the calculation we would like to make three observations:

First, our approximation does not yield a bound state for $N = 0$. It is therefore impossible to draw any rigorous conclusions about the zero energy state even though one could view the cancelling of the energy for the fast modes and the good (square

[10] Recall that the reduced mass is $m_r = 1/2\lambda$.

integrable at infinity) asymptotic behavior of the Green function of the Laplacian as mild evidence in favor of its existence.

A second point is that for $N \neq 0$ the linear potential allows for the existence of bound states within our approximation. Their energy scales like $\lambda^{1/3}$. We interpret this as a small increase over the BPS mass of the bound state $(2/\lambda)$ valid at small coupling. These states are therefore not BPS and most likely unstable in the full theory.

Finally notice that the "effective angular momentum" coming from the reduction of the problem to the equivalent radial problem is $L_{\rm eff} = 20$ and not $L_{\rm eff} = 12$ as one might guess by counting only the dimension of the moduli space $\mathbf{R}^9$. In this context it should be noted that this possible effect is quantum mechanical in nature and therefore consistent with the vanishing classical force calculated in [3] for D-particles at rest.

Now let us look more closely to the issue of gauge invariance of the wave function for $H_{\rm slow}$. At first, it might seem that one should take $l_2 = 0$, i.e., the wave function should be independent on n^a. In fact, due to the presence of the fermions, it is possible to allow for an n^a dependence by considering the combination $n^a\psi^a$. As shown in the previous section, at a generic point (λ_i, n^a) one can use 32 of the 48 fermions to make up the 16 creation and annihilation operators needed for $H_{\rm fast}$. This leaves 16 fermions, generically $n^a\psi^a_\alpha$ that can be used to make up the remaining 8 creation and annihilation operators needed to match the $24 = 27-3$ on shell bosonic degrees of freedom; they can be constructed as

$$\begin{aligned} a_{\hat\alpha} &= \frac{1}{\sqrt{2}} n^a(\psi^a_{\hat\alpha} + i\psi^a_{\hat\alpha+8}) \quad \text{for } \hat\alpha = 1, \cdots, 8 \\ a^\dagger_{\hat\alpha} &= (a_{\hat\alpha})^\dagger. \end{aligned} \tag{34}$$

We can act with up to $\tilde N_F = 8$ creation operators on the fermionic vacuum. The result of such an operation will not be directly an eigenstate of L_2^2 but it will contain all eigenstates $l_2 \leq \tilde N_F$, l_2 even (odd) if $\tilde N_F$ even (odd). However, each eigenstate can be easily projected out by taking the traceless components of product $n^{a_1} \cdots n^{a_{\tilde N_F}}$; e.g.

$$n^a n^b \to (n^a n^b - (1/3)\delta^{ab}) + (1/3)\delta^{ab} = \{l_2 \equiv 2\} \oplus \{l_2 \equiv 0\}. \tag{35}$$

By noticing that each allowed eigenvalue appears once in the reduction, we can calculate the degeneracies: For l_2 increasing from 0 to 8 we have: 128, 128, 127, 120, 99, 64, 29, 8 and 1. It is amusing to note that the state with the lowest degeneracy is the state with the highest value for l_2, an indication that the exact ground state might have a large l_2 component. In summary, the full wave function on $\mathcal{C}$ can be written as

$$\Xi = \mathcal{R}(r) Y_{l_8,\vec m}(\Omega_8) \mathcal{P}_{l_2}\left(a^\dagger_{\hat\alpha_1} \cdots a^\dagger_{\hat\alpha_{\tilde N_F}}\right)|0>, \tag{36}$$

where $\mathcal{P}_{l_2}$ is the projection described above, and $Y_{l_8,\vec m}$ are the "spherical harmonics"

on the eight-sphere[11]. By looking at (33) we see that $\mathcal{R} \to 0$ as $r \to 0$ unless $l_8 = l_2 = 0$, so the wave function is continuous at the origin as it should. Finally, in order for the wave function to be well defined after modding out the $\mathbf{Z}_2$ symmetry we must have $l_2 + l_8 =$ even integer.

The one particle equivalent problem (33) cannot be solved exactly but, if the internal quantum numbers are such that the potential is not too steep, its eigenvalues can be obtained through the WKB approximation in terms of a (non degenerate) radial quantum number n_r. After scaling out the dependence on g: $E_{\rm tot} = \lambda^{1/3}\epsilon$:

$$\int_{x_-}^{x_+} \sqrt{(\epsilon - V(x))} = \pi n_r \quad \text{where} \;\; V(x) = \frac{20 + l_8(l_8+7) + l_2(l_2+1)}{x^2} + Nx, \quad (37)$$

yielding, roughly, $\epsilon \approx (N n_r)^{2/3}$.

4 String interpretation

In treating the system of two D-branes as a dimensionally reduced $SU(2)$ Yang-Mills theory, solutions with gauge symmetry spontaneously broken to $U(1)$ describe branes which are separated from each other. Their separation can be read off in the masses of the charged excitations, given by the ground state energies of strings stretched between the two D-branes, and is proportional to the distance between them. Since spontaneous symmetry breaking only works in sufficiently high dimensions we expect quantum mechanical effects to modify the picture for lower dimensional branes. Arguments based on duality also indicate that D-particles in type IIA string theory should form (symmetric) bound states, one for each RR charge. We have seen that there are additional bound states of higher energies, at least in the Born-Oppenheimer approximation. We now wish to test how this spectrum of excited states can be understood in the string picture.

The Born-Oppenheimer approximation is adiabatic, which means that the slow modes are treated as static on the time-scale of the fast modes. Only after the effects of the fast modes has been taken care of does one study the motion of the slow modes. Before the dynamics of the slow modes has been taken into account, one essentially has the picture referred to above, with D-particles at fixed positions. To the system of two D-particles can be added any number of strings stretching between the two branes, or beginning and ending on the same brane. (Some linear combinations of states with strings beginning and ending on the same brane belong to the $U(1)$ describing the centre of mass motion of the pair of D-particles, and not to the $SU(2)$ of the relative motion, which we are focusing on here.)

In the low energy Yang-Mills approximation to the theory of open strings on the D-particles only the string ground states are taken into account. In our case they are simply the 8 vector components transverse to the straight strings stretching between

[11]For completeness, let us recall that the dimension of the first few such representations is 1, 9, 44, 156, 450 $\cdots$ for $l_8 = 0, 1, 2, 3, 4 \cdots$.

the two D-particles, in the bosonic sector, and the eight $SO(8)$ spinor components in the fermionic sector. For each of these modes the wave functions may have either positive or negative world-sheet parity, since the strings are oriented. (The Chan-Paton factor may be symmetric or anti-symmetric.) Any number of such strings may be excited[12]. We recognize the spectrum of fast modes in section 3.5, by interpreting N_i as the number of bosonic strings pointing in the ith transverse direction. The quantum number m_i counts the difference between the numbers of strings of positive and negative parity. Since there are only 16 different fermion states (no massive excitations in our approximation) there can be at most 16 fermionic strings stretching between the D-particles. Of course, bound states of two D-particles with an even number of fermionic strings will be bosons, but since the fermionic strings do not form bosonic bound states by themselves, we get this bound on the total number of fermionic strings stretching between the two branes.

Strings stretched between D-branes induce a linear potential between the branes, which causes them to move. The strength of the linear potential should be proportional to the number of strings connecting the D-particles. This is precisely what we see in the dynamics of the slow modes! We expect qualitatively similar effects in the dynamics of all Dirichlet p-branes with p spatial dimensions curled up around a compact manifold.

The Born-Oppenheimer approximation makes sense whenever the factorization in eq. (18) results in an approximate additivity of energies from fast and slow modes, i.e. the fast wave function $\mathbf{\Phi}$ should vary much slower with respect to the slow parameters than with respect to the fast variables. This happens for $Nr \gg \lambda/r^2$. The self-consistency of the approximation depends on the shape of the approximate wave function. As noted before the fast modes must be in an excited state, and in addition the wave function of the slow modes must be concentrated at $r \gg \lambda^{1/3}$. This is most easily achieved by taking the angular momentum l_8 to satisfy $l_8^2 \gg 1$.

The bound states that we have found in the Born-Oppenheimer approximation have positive energies, and should be able to decay. Indeed, a four point coupling between two massive W particles and two photons in the Yang-Mills theory corresponds to an amplitude of order λ between two strings stretching between fixed D-particles and two strings, one fixed to each of the two stationary D-particles. In the present Born-Oppenheimer framework part of these interactions are already included and generate the motion of the D-particles, but we suspect that one would also see decay processes where two stretched strings annihilate and their energy goes into kinetic energy of the D-particles, if one improves on the approximations.

[12]For n D-particles there are $8n(n-1)$ ways for bosonic straight strings to connect two D-particles, in agreement with the counting of bosonic non-degenerate directions for $G = SU(n)$ at the end of section 3.4.

5 Conclusions

We have shown, by studying the low energy effective action for two D-particles, how it is possible to extract information about bound states above the BPS threshold. These states have a rather simple interpretation as being generated by strings stretching between the D-particles. Since these states are not BPS they will most likely decay in the full theory and therefore should only be interpreted as metastable. It should be possible to study their decay within the black hole picture. In this regard we must keep in mind that the degeneracies obtained are only power law, as they arise from a finite number of oscillators. They might contribute to subleading corrections to the black hole entropy.

It is also amusing to write down the expression for the mass of the system using the metric appropriate for eleven dimensional supergravity. This rescales the masses by a factor $\lambda^{1/3}$. If we further use the identification of [13] where $R = \lambda^{2/3}$ we find that

$$M = 2/R + R\epsilon \tag{38}$$

where R is the compactification radius in going from eleven dimensions to ten and ϵ is the eigenvalue in (37). (Note that this is only valid for R small.) The simple dependence on the compactification radius suggests that these states might have a simple eleven dimensional explanation.

Note added

After this paper was submitted a closely related paper by D. Kabat and P. Pouliot appeared, hep-th/9603127.

Acknowledgements.

We wish to thank A. Alekseev, J. Kalkkinen, A. Niemi, B. Nilsson, I. Pesando, H. Rubinstein and K. Sfetsos for helpful discussions.

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